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Short Communication

Remarks on robust stability of saturation controllers

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Abstract

In active structural vibration control, actuator's saturation and robust stability with respect to parameter uncertainties are practical and important issues. Saturated sliding mode controller and robust saturation controller, which were presented in previous researches, are suitable in this case. In this paper, robust stabilities of the two controllers are examined and compared through numerical simulations for a 2dof vibrating system.
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1. Introduction

In active structural vibration control, controller design, which involves both actuator's saturation and parameter uncertainties, is needed. Large structures as ships and high-rise buildings are very massive and very large active control input force, which is often unrealistic, is demanded correspondingly. And as the structures are bigger and higher or more complex, it is more difficult to know exact values of their masses and stiffnesses. Therefore, in controller design for active structural vibration control, actuator's saturation needs to be considered and the controller should be designed to be robust with respect to parameter uncertainties.

There have been many researches on controller design including only actuator's saturation in the field of active structural vibration control [1–6]. Even though these controllers guarantee stability in nominal system with actuator's saturation, they cannot do robust stability in uncertain system with parameter uncertainties. The controller, which does not consider parameter uncertainties, may lose stability in uncertain system with parameter uncertainties. Therefore, controllers considering both actuator's saturation and parameter uncertainties have been presented to serve the purpose for robust stability. Yang et al. [7,8] presented a saturated sliding mode controller (SSMC) based on the theory of the traditional sliding mode control (SMC) and proved it to be effective method in active structural vibration control. Even though this controller is robust with respect to parameter uncertainties of system, it cannot prescribe bounds of parameter uncertainties of system within which closed-loop robust stability is guaranteed certainly. On the other hand, Lim et al. [9] developed a robust saturation controller (RSC) for linear time-invariant system based on the affine quadratic stability definition and multi-convexity concept and showed the availability of this controller

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for linear vibrating system. This RSC can analytically address bounds of parameter uncertainties within which closed-loop robust stability is guaranteed.

In the previous paper [9], control performances for the SSMC and the RSC are compared through numerical simulations for a 3dof linear vibrating system. In this paper, robust stabilities of the SSMC and the RSC are examined and compared more precisely through numerical simulations. A 2dof linear vibrating system is selected and several numerical simulation results for the two controllers are shown.

2. Saturated sliding mode controller

In the SSMC, the following nominal linear time-invariant system (1) is considered and the controller is designed based on the theory of the traditional SMC

$$\dot{x}(t) = A_0 x(t) + Bu(t), \quad x(0) = x_0, \tag{1}$$

where $x = [x_1\, x_2 \ldots x_n]^{\mathrm{T}}$ is an $n \times 1$ state vector, A_0 is an $n \times n$ nominal system matrix and is assumed to be stable, B is an $n \times 1$ control input vector, and u is a scalar control input with control input constraint (2)

$$|u(t)| \leq u_{\max}. \tag{2}$$

The theory of the SMC is to design controllers to drive the system's response into the sliding surface on which the motion is stable. Let $s = 0$ be the sliding surface

$$s = P_s x = 0, \tag{3}$$

where P_s is an $1 \times n$ vector to be determined such that the motion on the sliding surface is stable. The method of linear quadratic regulator is used to determine P_s [8]. P_s is obtained by minimizing J_s as follows:

$$J_s = \int_0^{\infty} x^{\mathrm{T}} Q_s x \, \mathrm{d}t, \tag{4}$$

where Q_s is an $n \times n$ positive-definite matrix.

The controllers are designed to drive the state trajectory into the sliding surface $s = 0$. To achieve this goal, the following Lyapunov function is considered:

$$V_s(x) = \tfrac{1}{2} s^{\mathrm{T}} s = \tfrac{1}{2} x^{\mathrm{T}} P_s^{\mathrm{T}} P_s x. \tag{5}$$

The sufficient condition for the sliding mode $s = 0$ to occur as time goes to ∞ is $\dot{V}_s(x) = s^{\mathrm{T}} \dot{s} < 0$. Taking the time derivative and using state equation (1), one obtains the following:

$$\dot{V}_s(x) = x^{\mathrm{T}} P_s^{\mathrm{T}} P_s A_0 x + x^{\mathrm{T}} P_s^{\mathrm{T}} P_s B u. \tag{6}$$

For $\dot{V}_s(x) < 0$ with control input constraint (2), the following controller of saturation type was suggested by Yang et al. [7,8]:

$$u(t) = -\mathrm{sat}[(\alpha_s (P_s B)^{-1} P_s A_0 + \delta_s B^{\mathrm{T}} P_s^{\mathrm{T}} P_s) x(t)], \tag{7}$$

where $0 \leqslant \alpha_s \leqslant 1$ and $\delta_s > 0$ is referred to as the sliding margin.

Robust stability of the SSMC (7) is explained only using the property of the robustness of the SMC with respect to parameter uncertainties. Even though the SMC is robust with respect to parameter uncertainties of system, it cannot prescribe bounds of parameter uncertainties of system within which closed-loop robust stability is guaranteed certainly. The complete response of a SMC system consists of two phases of different modes: the reaching mode and the sliding mode. Robustness of the SMC with respect to parameter uncertainties is guaranteed only in the sliding mode. Therefore, the robustness of the SMC is not guaranteed over the complete response of a SMC system. Furthermore, system's response applying the SSMC remains in the reaching mode for much time than unsaturated system applying the SMC and bounds of parameter uncertainties within which robust stability is guaranteed are narrower than unsaturated SMC system. This will be shown through numerical simulations later.

3. Robust saturation controller

In the RSC, the following uncertain linear time-invariant system (9) is considered and the controller is designed based on the affine quadratic stability definition and multi-convexity concept

$$\dot{x}(t) = (A_0 + \Delta A(\theta))x(t) + Bu(t), \quad x(0) = x_0, \tag{9}$$

where $x = [x_1\, x_2 \ldots x_n]^{\mathrm{T}}$ is an $n \times 1$ state-vector, A_0 is an $n \times n$ nominal system matrix, $\theta = (\theta_1, \theta_2, \ldots, \theta_k) \in \Re^k$ is a vector of uncertain real parameters, $\Delta A(\theta)$ is time-invariant uncertainties, $A_0 + \Delta A(\theta)$ is assumed to be stable, B is an $n \times 1$ control input vector, and u is a scalar control input with control input constraint (2).

We assume that lower and upper bounds are available for the parameter values. Specifically, each parameter θ_i ranges between known external values $\underline{\theta_i}$ and $\overline{\theta_i}$

$$\theta_i \in [\underline{\theta_i}, \overline{\theta_i}] \quad \text{for } i = 1, 2, \ldots, k. \tag{10}$$

This means that the parameter vector θ is valued in a hyper-rectangle called the parameter box. In the sequel

$$\Theta := \{(\omega_1, \omega_2, \ldots, \omega_k) : \quad \omega_i \in \{\underline{\theta_i}, \overline{\theta_i}\}\} \tag{11}$$

denotes the set of the 2^k vertices or corners of these parameters.

The uncertain system matrix $A(\theta)$ depends affinely on the uncertain parameters of θ_i and is described by the system with structured real parameter uncertainties. That is

$$A(\theta) = A_0 + \Delta A(\theta) = A_0 + \theta_1 A_1 + \theta_2 A_2 + \cdots + \theta_k A_k, \tag{12}$$

where $A_0, A_1, A_2, \ldots, A_k$ are known fixed matrices.

The following notion of parameter-dependent quadratic Lyapunov function is defined

$$V(x, \theta) = x^{\mathrm{T}} P(\theta) x, \tag{13}$$

where $P(\theta)$ is a symmetric positive-definite matrix and is an affine function of θ

$$P(\theta) = P_0 + \theta_1 P_1 + \theta_2 P_2 + \cdots + \theta_k P_k. \tag{14}$$

The time derivative of the Lyapunov function (13) is of the following:

$$\dot{V}(x, \theta) = x^{\mathrm{T}}[A(\theta)^{\mathrm{T}} P(\theta) + P(\theta)A(\theta)]x + 2x^{\mathrm{T}} P(\theta) Bu. \tag{15}$$

For $\dot{V}(x, \theta) < 0$ with control input constraint (2), the following controller of saturation type was proposed by Lim et al. [9]

$$u(t) = -\mathrm{sat}[\delta B^{\mathrm{T}} P_0 x(t)], \tag{16}$$

where $k+1$ symmetric matrices $P_0, P_1, P_2, \ldots, P_k$, and symmetric positive-definite matrix M_a satisfy $2^{k+1}+k$ LMI conditions of Eqs. (17)–(19) and $\delta > 0$ satisfies Eq. (20) for these matrices $P_0, P_1, P_2, \ldots, P_k$ and M_a.

$$P(\omega) > 0 \quad \text{for all } \omega \in \Theta, \tag{17}$$

$$A(\omega)^{\mathrm{T}} P(\omega) + P(\omega)A(\omega) + M_a < 0 \quad \text{for all } \omega \in \Theta, \tag{18}$$

$$A_i^{\mathrm{T}} P_i + P_i A_i \geqslant 0 \quad \text{for } i = 1, 2, \ldots, k, \tag{19}$$

$$M_a + \delta\{2P_0 BB^{\mathrm{T}} P_0 + \sum_{i=1}^{k} \theta_i (P_0 BB^{\mathrm{T}} P_i + P_i BB^{\mathrm{T}} P_0)\} > 0 \quad \text{for all } \omega \in \Theta. \tag{20}$$

Note that M_a is a controller design parameter and $\delta_{\max}$, maximum value of δ, satisfying Eq. (20) is finite in uncertain system. LMIs of Eqs. (17)–(20) can be easily solved using commercial Matlab® and LMI control toolbox [10].

Unlike the SSMC (7), the RSC (16) is robustly stable with respect to parameter uncertainties over the prescribed upper and lower bounds because bounds of parameter uncertainties are considered analytically in the design of this controller.

4. Numerical simulations

In the following numerical simulations, robust stabilities of the SSMC (7) and the RSC (16) are examined and compared. A 2dof linear vibrating system with one control input force is considered as shown in Fig. 1. The masses, stiffnesses, and damping coefficients for nominal system are $m_1 = m_2 = 1\,\text{kg}$, $k_1 = k_2 = 1\,\text{N/m}$, and $c_1 = c_2 = 0.01\,\text{N s/m}$, respectively. The maximum control input force $u_{\max} = 1\,\text{N}$. Let uncertainties of stiffnesses be θ_1 and θ_2, then the admissible trajectories are given by $k_1(1+\theta_1)$ and $k_2(1+\theta_2)$ specified in multiplicative form and this uncertain system can be described by state-space equation as in Eq. (9). In this case, state vector $x = [x_1\, x_2\, \dot{x}_1\, \dot{x}_2]^{\mathrm{T}}$, control input vector $B = [0\, 0\, 1/m_1\, 0]^{\mathrm{T}}$, and uncertain system matrix $A(\theta)$ is described by Eq. (21)

$$A(\theta) = A_0 + \theta_1 A_1 + \theta_2 A_2, \tag{21}$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & -\dfrac{c_1+c_2}{m_1} & \dfrac{c_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & \dfrac{c_2}{m_2} & -\dfrac{c_2}{m_2} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{k_1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{k_2}{m_1} & \dfrac{k_2}{m_1} & 0 & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & 0 & 0 \end{bmatrix}.$$

Simulation results for the case with parameter uncertainties of $k_1(1-\theta_s)$ and $k_2(1+\theta_s)$ are presented under initial condition $x_0 = [0\,0\,0\,6]^{\mathrm{T}}$.

Robust stability of the SSMC is firstly examined. The method of linear quadratic regulator is used for the design of the sliding surface with a diagonal weighting matrix $Q_s = \mathrm{diag}(10, 10, 1, 1)$. This results in a sliding surface $s = P_s x = 4.0301x_1 + 0.4421x_2 + 1.0\dot{x}_1 + 3.2481\dot{x}_2 = 0$. The SSMC is designed using $\delta_s = 10$ and $\alpha_s = 1$. Fig. 2 shows displacement of x_1 and control input force for nominal system applying the SSMC. As shown in Fig. 2, stability of the SSMC for nominal system is guaranteed analytically when saturation of

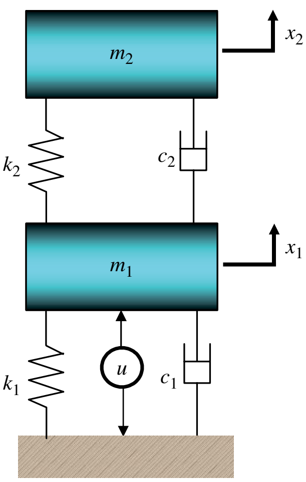

Fig. 1. 2dof vibrating system with one actuator.

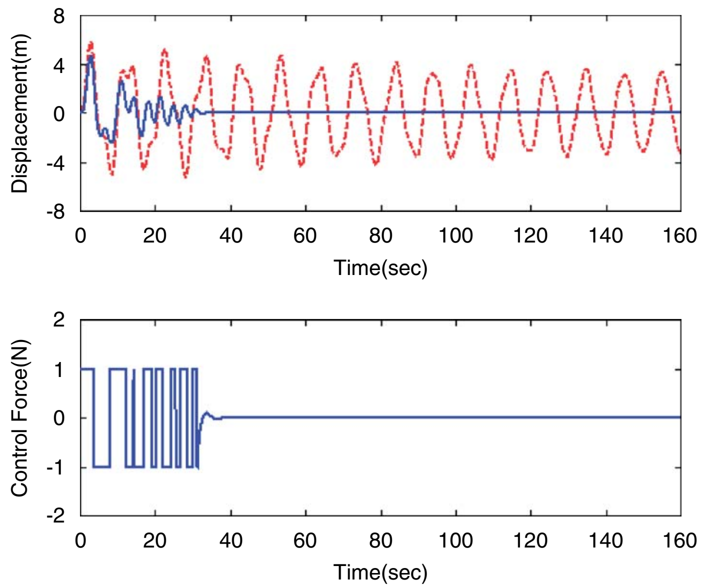

Fig. 2. Displacement and control input force for nominal system applying the saturated sliding mode controller under $x_0 = [0\,0\,0\,6]^T$ (- - - -, No control; —, SSMC).

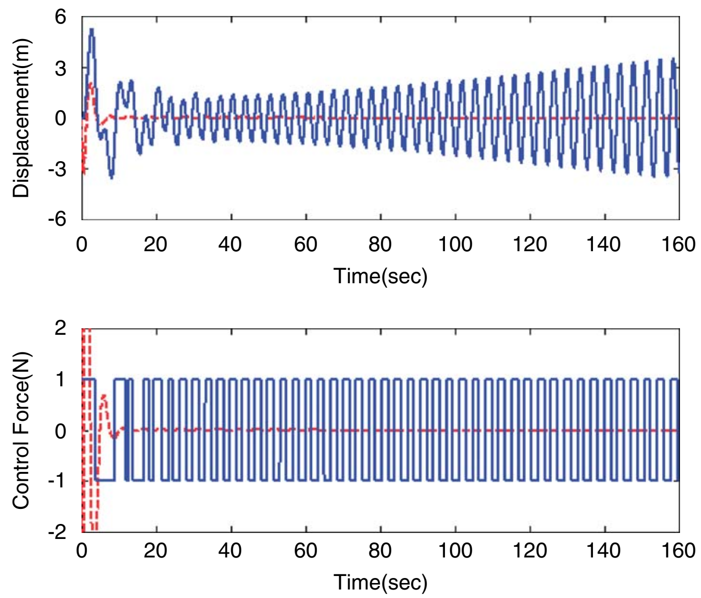

Fig. 3. Displacements and control input forces for uncertain system with $\theta_s = 0.2$ applying the sliding mode controller and the saturated sliding mode controller under $x_0 = [0\,0\,0\,6]^T$ (- - - -, SMC w/o saturation; —, SSMC with saturation).

control input force occurs. In case of the SMC without saturation of control input force, bounds of parameter uncertainties within which robust stability is guaranteed are very wide. But in case of the SSMC, bounds of parameter uncertainties within which robust stability is guaranteed are narrow. It is checked through numerical simulations that this SSMC is unstable when θ_s is greater than 0.2. Fig. 3 compares displacements of x_1 and control input forces for the uncertain system with $\theta_s = 0.2$ applying the SMC and the SSMC. Fig. 4 compares sliding surfaces ($s = P_s x$) of the two controllers for this case. The SMC without saturation of control input force is stable even in this case because using a big control input force without saturation makes

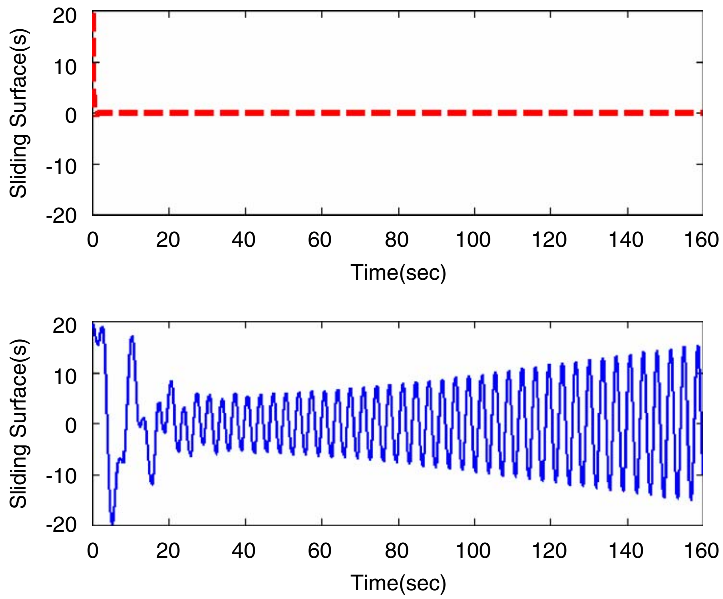

Fig. 4. Sliding surfaces for uncertain system with $\theta_s = 0.2$ applying the sliding mode controller and the saturated sliding mode controller under $x_0 = [0\,0\,0\,6]^T$ (- - - -, SMC w/o saturation; —, SSMC with saturation).

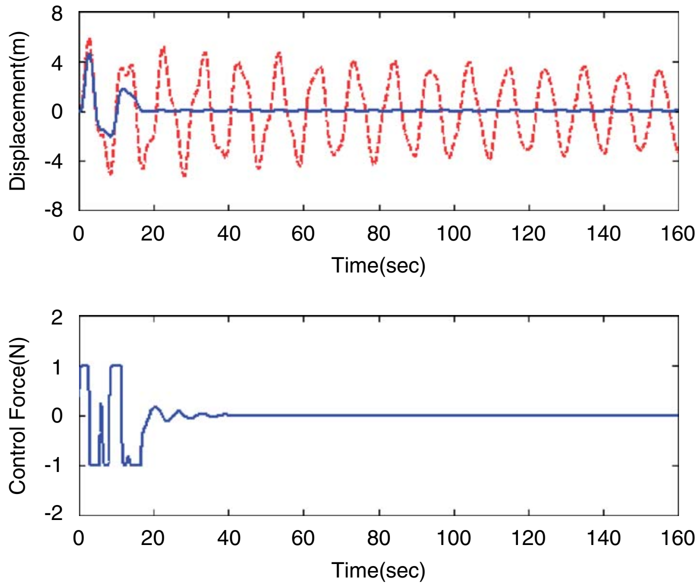

Fig. 5. Displacement and control input force for nominal system applying the robust saturation controller under $x_0 = [0\,0\,0\,6]^T$ (- - - -, No control; —, RSC).

the system's response reach fast the sliding mode as shown in Fig. 4. On the other hand, the SSMC with saturation of control input force is unstable in this case. Figs. 3 and 4 show that when parameter uncertainties exist, robust stability of the SSMC may be not guaranteed because using control input force with saturation makes time when system's response reaches the sliding mode longer.

Next, robust stability of the RSC is checked. Let $|\theta_1| \leqslant 0.9$ and $|\theta_2| \leqslant 0.9$, which are somewhat wide, for two parameter uncertainties of stiffnesses. The controller design parameter $M_a = 5\text{e} - 4\,\text{diag}(1,1,1,1)$ is chosen. The computed value of $\delta_{\max}$ is about 6.84e−2. The control gain of designed RSC is

$\delta_{\max}B^{\mathrm{T}}P_0 = [-1.6759\mathrm{e}-2 \quad 5.6598\mathrm{e}-3 \quad -3.9096 \quad -1.9120\mathrm{e}-4]$. Figs. 5–7 show displacement of x_1 and control input force for nominal system, for the uncertain system with $\theta_s = 0.2$, and for the uncertain system with $\theta_s = 0.9$, respectively, applying the RSC. Fig. 6 shows that the RSC guarantees robust stability for the uncertain system with parameter uncertainties of $\theta_s = 0.2$. It is also ascertained that the RSC guarantees robust stability within all the range of parameter uncertainties considered in controller design. It can be seen from Fig. 7 that the RSC guarantees robust stability for the uncertain system with parameter uncertainties of

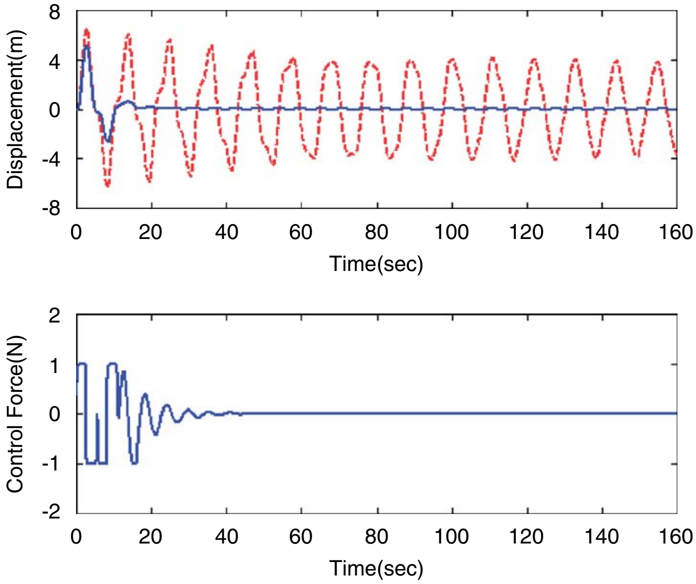

Fig. 6. Displacement and control input force for uncertain system with $\theta_s = 0.2$ applying the robust saturation controller under $x_0 = [0\,0\,0\,6]^{\mathrm{T}}$ (- - - -, No control; —, RSC).

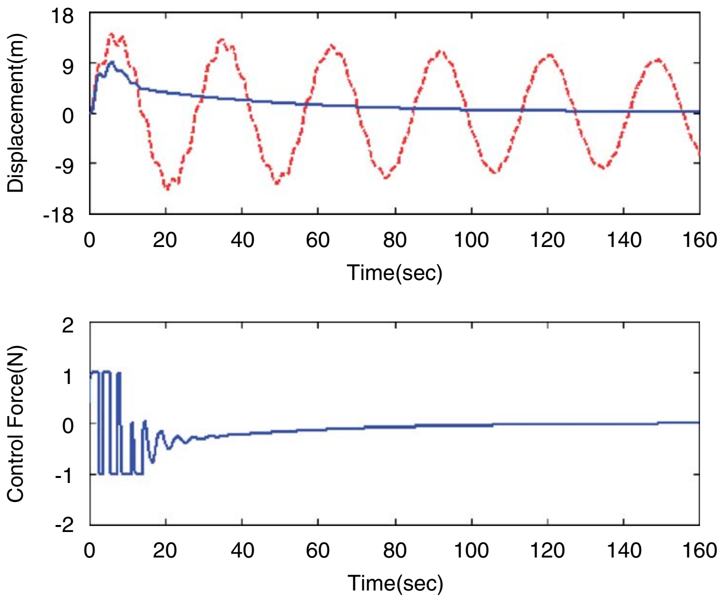

Fig. 7. Displacement and control input force for uncertain system with $\theta_s = 0.9$ applying the robust saturation controller under $x_0 = [0\,0\,0\,6]^{\mathrm{T}}$ (- - - -, No control; —, RSC).

$\theta_s = 0.9$. Simulation results show that the RSC is robustly stable with respect to parameter uncertainties over the prescribed upper and lower bounds.

Numerical simulation results for a variety of initial conditions show that the stability margin of the SSMC for uncertain system depends on initial condition of the system. Under initial conditions like $x_0 = [0\,0\,0\,7.4]^T$ and $x_0 = [1.9\,7\,0\,0]^T$, the SSMC makes the system unstable when θ_s is greater than 0.1. And under initial conditions like $x_0 = [0\,0\,0\,6]^T$, $x_0 = [2\,6\,0\,0]^T$, and $x_0 = [3\,2\,0\,0]^T$, the SSMC is unstable when θ_s is greater than 0.2. Moreover, under initial conditions like $x_0 = [2\,5\,0\,0]^T$, $x_0 = [2.9\,6\,0\,0]^T$, and $x_0 = [0\,0\,0\,4.4]^T$, robust

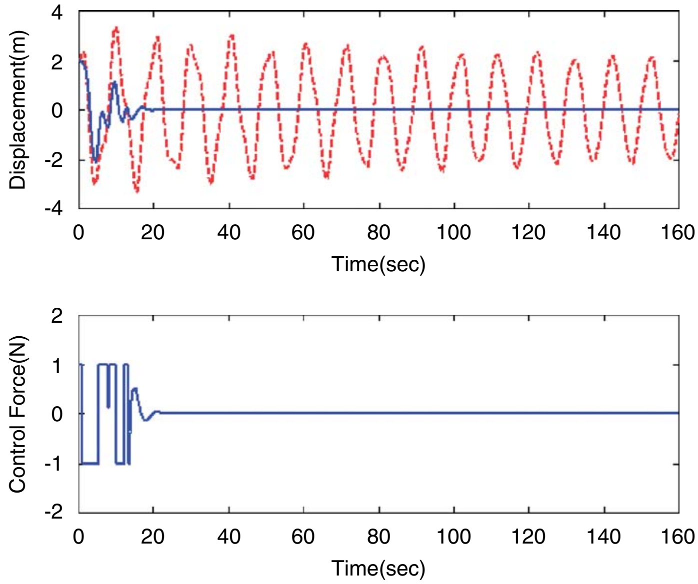

Fig. 8. Displacement and control input force for nominal system applying the saturated sliding mode controller under $x_0 = [2\,5\,0\,0]^T$ (- - - -, No control; —, SSMC).

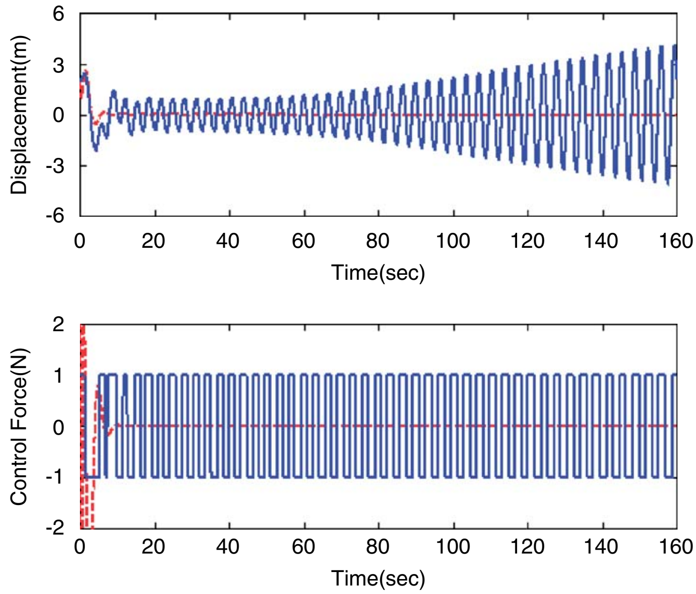

Fig. 9. Displacements and control input forces for uncertain system with $\theta_s = 0.3$ applying the sliding mode controller and the saturated sliding mode controller under $x_0 = [2\,5\,0\,0]^T$ (- - - -, SMC w/o saturation; —, SSMC with saturation).

stability of the SSMC is not guaranteed when θ_s is greater than 0.3. As an additional simulation result, Figs. 8 and 9 show displacement of x_1 and control input force for nominal system and for the uncertain system with $\theta_s = 0.3$, respectively, applying the SSMC under $x_0 = [2\,5\,0\,0]^T$. But the RSC is always stable under all initial conditions within the prescribed bounds of parameter uncertainties because they are considered in controller design. For comparison, Figs. 10 and 11 show displacement of x_1 and control input force for nominal system and for the uncertain system with $\theta_s = 0.3$, respectively, applying the RSC under $x_0 = [2\,5\,0\,0]^T$.

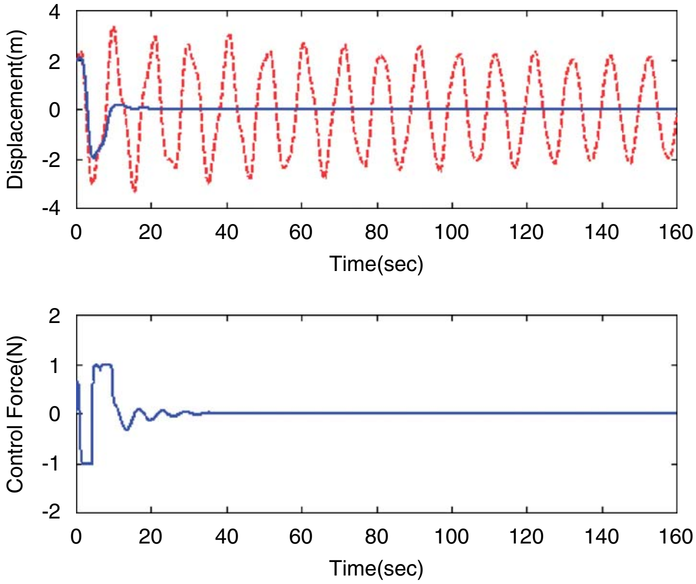

Fig. 10. Displacement and control input force for nominal system applying the robust saturation controller under $x_0 = [2\,5\,0\,0]^T$ (- - - -, No control; —, RSC).

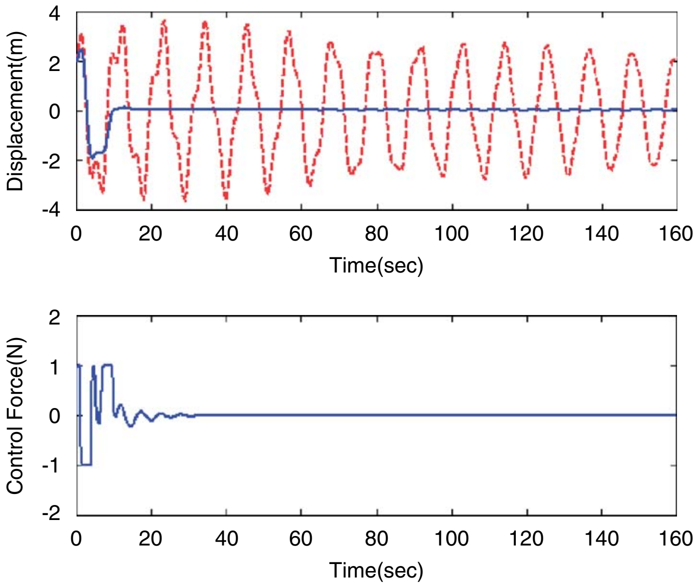

Fig. 11. Displacement and control input force for uncertain system with $\theta_s = 0.3$ applying the robust saturation controller under $x_0 = [2\,5\,0\,0]^T$ (- - - -, No control; —, RSC).

5. Conclusions

In this paper, robust stabilities of the saturated sliding mode controller (SSMC) (7) and the robust saturation controller (RSC) (16) are examined and compared through numerical simulations.

The RSC always guarantees robust stability with respect to parameter uncertainties over the prescribed upper and lower bounds when saturation of control input occurs because bounds of parameter uncertainties are considered analytically in the design of this controller.

On the other hand, the SSMC is designed using only nominal system and robust stability of the controller is explained using the property of the robustness of the SMC with respect to parameter uncertainties. This controller is always stable for nominal system when saturation of control input occurs. But bounds of parameter uncertainties within which robust stability is guaranteed can be checked by only numerical simulations. And system's response applying the SSMC remains in the reaching mode for much time than unsaturated system applying the SMC and bounds of parameter uncertainties within which robust stability is guaranteed are narrower than unsaturated SMC system. Therefore, research on the SSMC in which bounds of parameter uncertainties are considered analytically in the design of the controller to guarantee robust stability of the controller over the complete response including both the reaching mode and the sliding mode is needed.

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